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QUANTUM MECHANICS AS RELATIVISTIC STATISTICS (THE CASE OF INTERACTING PARTICLES)

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ABSTRACT: It is shown that non-relativistic quantum mechanics of two particles interacting with an external electromagnetic field and with each other can be considered as the statistics of two-dimensional surfaces representing the state of the system consisting of two non-deterministic particles in eight-dimensional space, which is the direct product of the space-times for each particle.

The concept advanced in [1, 2] is applied in this work to the case of two /5*
particles interacting with an electromagnetic field and with each other[1]. Quantum mechanics [1, 2] is a form of relativistic statistics. It will be shown in this work, in particular, that the quantum mechanics of two interacting particles can be represented as the statistics of two-dimensional surfaces representing the r-state[2] of two particles in eight-dimensional space V_{12}, which is the direct product of the space-time V_1 and V_2 for each particle.

We will formulate the fundamental concept. Classical particles[3] presumably /6
interact non-relativistically with the environment (space). Consequently their behavior becomes non-deterministic and unpredictable. The behavior of particles can be described only statistically. The statistical principle [2] is used for this purpose. With the aid of this principle a deterministic system —

*Numbers in the margin indicate pagination in the foreign text.

[1]A bibliography and review of works on interpretation of quantum mechanics from the viewpoint of classical mechanics can be found in [3].

[2]Two different concepts of the state of the system are used in the work. The n-state (non-relativistic state) is defined at a given moment of time. Evolution of the n-state is described by equations of motion. The n-state of a particle is its coordinates and pulse.

[3]The r-state (relativistic state) is defined in the entire space-time. The r-state obeys certain equations, which act as restrictions imposed on the permissible r-states. The r-states of a particle is the equation of its world line $q^i = q^i(\tau)$. See [1] for greater detail.

[3]Particles are classical in the sense that the motion of each of them can be described by the world line in the space-time.

statistical set — can be made to correspond to the non-deterministic system. The relation between the statistical set and corresponding non-deterministic system is established on the basis of the following two properties.

1. The state of a set is the density of the state of the systems comprising it.

2. Any additive value ascribed to the set as a dynamic system (energy, motion, etc.) is (with the appropriate standardization of the set) the mean value for the systems comprising the set.

It turns out that it is possible to select the Lagrangian for a statistical set, such that the description in the non-relativistic limit will be equivalent to the quantum mechanics description. In order to achieve equivalent with quantum mechanics it is important that the particles be described with the aid of the relativistic concept of state (r-state).

I. Statistical Set for Two Particles in Electromagnetic Field /7

We will discuss a system consisting two particles. The r-state of the eighth particle is described by the world line L_A : $q^i_A = q^i_A\ (\tau_A)$, i = 0, 1, 2, 3, A = 1, 2 in space-time V_A (q^i_A [i = 0, 1, 2, 3] are the coordinates in space capsule V_A). The r-state of a system of two particles is described by two-dimensional surface $S = L_1 \otimes L_2$ in eight-dimensional space $V_{12} = V_1 \otimes V_2$. The symbol $\otimes$ denotes the direct product.

We introduce the coordinates $\otimes$ (a = 1, 2, ...8) in V_{12}:

$$x^a = q^i_A = q^{(i)}, \quad a = 4(A - 1) + i + 1. \tag{1.1}$$

In the ensuing discussion we will use, in addition to tensorial indices a, b,... the double index $\binom{i}{A}$. The correspondence between them is established by the relation

$$a \leftrightarrow \binom{i}{A}, \quad a = 4\ (A - 1) + i + 1, \tag{1.2}$$

$$a = 1, 2, \ldots 8, \ i = 0, 1, 2, 3 \quad A = 1, 2$$

Summation is done in terms a, b,... from 1 to 8 and in terms of i, j,... from 0 to 3 according to the recurrence of Latin tensorial indices, and from 1 to 3 according to the Greek indices. Summation in terms of the capital indices, indicating the number of the particle, is always denoted by the summation symbol.

As shown in [2], the density of surfaces of state S in the vicinity of point x of space V_{12} is determined by skew-symmetric tensor $j^{ab}(x)$. In the case at hand, when $S = L_1 \otimes L_2$,

$$j^{\binom{i}{1}\binom{K}{1}} = j^{\binom{i}{2}\binom{K}{2}} = 0, \; i, K = 0, 1, 2, 3. \tag{1.3}$$

According to the statistical principle [2], the density j^{ab} of surfaces of state S is the state of a statistical set of two-particle systems. We will call this set a quantum set. In the case when the particles interact only with an external electromagnetic field, the action for it can be written in the form /8

$$S = S_{cl} + S_{qu}, \tag{1.4}$$

$$S_{cl} = S_{cl}[j^{ab}, pa, \xi^{\beta}_{\alpha}] = \int \Big\{ \sum_{A=1}^{2} m_A \frac{j^{\binom{\alpha}{A}b}\eta_b \, j^{\binom{\alpha}{A}c}\eta_c}{s j^{\binom{0}{A}d}\eta_d} - \tag{1.5}$$

$$- p_a \eta_b \left(j^{ab} - \frac{\partial^2 j}{\partial \tau_a \partial \eta_b} \right) + \frac{e_A}{c} j^{\binom{i}{A}b} \eta_b A_{\binom{i}{A}}(q_A) \Big\} \delta(u - C) d^8x,$$

$$S_{qu} = S_{cl}\left[j^{ab} \right] = - \int \sum_{A=1}^{2} \frac{\hbar^2}{8M_A} \frac{\dfrac{\partial j^{\binom{0}{A}b}}{\partial q^{\alpha}_{A}} \eta_b \dfrac{\partial j^{\binom{0}{A}c}}{\partial q^{\alpha}_{A}} \eta_c}{j^{\binom{0}{A}d} \eta_d} \delta(\eta - C) d^8x, \tag{1.7}$$

where $\vec{\eta} = \vec{\eta}(t_1, t_2)$, $t_1 = q_1^0$, $q_A = \{q_A^0, q_A^1, q_A^2, q_A^3\}$.

$\eta = C =$ const is to a certain extent an arbitrary 7-dimensional surface in space V_{12}, in terms of which integration is done. The values j^{ab}, a, $\overset{B}{\xi}_{\alpha}$, $B = 1, 2, \ldots 2S$ are the variables to be changed, e_A, m_A are the charge and mass, respectively, of the A-th particle and c is the velocity of light:

$$j = \sum_{B=1}^{S} \frac{\partial\left(\tau, \overset{2B-1}{\xi}_1, \overset{2B-1}{\xi}_2, \overset{2B-1}{\xi}_3, \eta, \overset{2B}{\xi}_1, \overset{2B}{\xi}_2, \overset{2B}{\xi}_3\right)}{\partial(\chi^1, \chi^2, \chi^3, \chi^4, \chi^5, \chi^6, \chi^7, \chi^8)}, \tag{1.8}$$

$$\tau_a \equiv \frac{\partial\tau}{\partial\chi^a}, \quad \eta_b \equiv \frac{\partial\eta}{\partial\chi^b}$$

$\dfrac{\partial^2 j}{\partial\tau_a \partial\eta_b}$ is the partial derivative of J in terms of τ_a and η_b for fixed $\overset{\beta}{\xi}_{\alpha,a} \equiv$ $\equiv \partial\overset{\beta}{\xi}_{\alpha}/\partial\chi^a$. $A_{\underset{(A)}{i}}(q_A)$ is the fore-potential of the external electromagnetic field in space V_A. Since the electromagnetic field is external

$$A_{\underset{(1)}{i}}(q) = A_{\underset{(2)}{i}}(q) = A_i(q), \tag{1.9}$$ /9

i.e., the form of functions $A_{\underset{(1)}{i}}$ and $A_{\underset{(2)}{i}}$ is identical and they depend on different arguments in accordance with the fact that they pertain to different spaces V_A.

j^{ab}, as we already mentioned, is the density of surfaces of state S. $S \cdot \rho_a = \rho_{\underset{(A)}{i}}(x)$ is the canonical pulse, i.e., the mean of canonical pulse p_i

of the A-th particle at point x. Formally p_a is the Lagrange factor, introducing the definition

$$j^{ab}\eta_b = \frac{\partial j}{\partial \tau_a} \tag{1.10}$$

It should be borne in mind that expression (1.6) for action S_{cl} can be derived from theaaction for two particles in the external magnetic field:

$$S = \sum_{A=1}^{2} \int \left(-m_A C \sqrt{dq_A^i \, g_{ik} dq_A^K} + \frac{e_A}{c} A_i(q_A) dq_A^i \right), \tag{1.11}$$

$$g_{ik} = \begin{Vmatrix} C^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \tag{1.12}$$

This can be done by examining a simple set[4] of deterministic systems described by action (1.11). The derivation can be done with the aid of the method employed in [2] for free particles. It turns out that ξ^1_α and ξ^2_α denote the /10
Lagrangian coordinates, i.e., the set of six values ξ^1_α ξ^2_α $\alpha = 1, 2, 3$ defines the number of the system in the set. Thus, the writing of S_{cl} in form (1.6) is not an arbitrary construction. It is noteworthy that examination of simple set inevitably leads to S = 1 in (1.8). Furthermore, η is an arbitrary function not only of t_1, t_2, but of x. In this sense discussion of a nonsimple set and the discarding of the condition S = 1 in (1.8) are a generalization of

[4] A simple set is defined as the set of surfaces of state S in which surfaces S do not intersect (see [2]).

the result of (1.11). We will not discuss here the need for introducing the non-simple set. This is examined in [2].

The introduction of action (1.7), in contrast to (1.6), is a special assumption. (1.7) takes into account the non-determinancy of motion of a separate system. The result of this non-determinancy is "diffusion" of surfaces S from region of space to another. The rate of this process is proportional to the gradient of density j^{ab} of states of the systems. Accordingly, the Lagrangian is proportional to the square of density gradient j^{ab}, and the proportionality factor is $\hbar^2/8M_A$ (t is the Planck constant). S_{qu} essentially contains the gradient of only one component $j^{\binom{0}{1}\binom{0}{2}}$, since in the non-relativistic case at hand all other components are much less than $j^{\binom{0}{1}\binom{0}{2}} = -j^{\binom{0}{2}\binom{0}{1}}$. Thus (1.17) is a special assumption, the validity of which should be borne out by the results.

We will now consider that in view of $\eta = \eta(t_1,t_2)$

$$\eta_{\binom{\alpha}{A}} = \frac{\partial\eta}{\partial q^{\alpha}_{A}} = 0, \qquad \alpha = 1,\ 2,\ 3,\ A = 1,\ 2, \tag{1.13}$$

and with the aid of (1.3) we will write the action in the form

$$S = S_m + S_{m\gamma} + S_{qu}, \tag{1.14}$$ /11

$$S_m = S_m\left[j^{ab},\ p_a,\ \zeta^{\beta}_{\alpha}\right] = \int \sum_{A=1}^{2} \left\{ m_A \frac{j^{\binom{\alpha}{A}\binom{0}{3-A}}\, j^{\binom{\alpha}{A}\binom{0}{3-A}}}{s j^{\binom{0}{A}\binom{0}{3-A}}}\, \eta_{\binom{0}{3-A}} - \sum_{B=1}^{2} p_{\binom{i}{A}}\, \eta_{\binom{0}{B}} \left((1-\delta_{AB})\, j^{\binom{i}{A}\binom{0}{3-A}} - \frac{\partial^2 j}{\partial\tau_{\binom{i}{A}}\,\partial\eta_{\binom{0}{B}}} \right) \right\} \delta(\eta - C)\, d^8x \tag{1.15}$$

$$S_{m\gamma} = S_{m\gamma}\left[j^{ab}\right] = \sum_{A=1}^{2}\int \frac{e_A}{c}\, j^{\binom{i}{A}\binom{0}{3-A}}\, \eta_{\binom{0}{3-A}}\, A_i(q_A)\, \delta(\eta - C)\, d^B x, \tag{1.16}$$

$$S_{gu} = S_{gu}\left[j^{ab}\right] = -\int\sum \frac{\hbar^2}{8m_A} \frac{\dfrac{\partial j^{\binom{A}{0}\binom{0}{3-A}}}{\partial q_A^\alpha}\,\dfrac{\partial j^{\binom{0}{A}\binom{0}{3-A}}}{\partial q_A^\alpha}}{j^{\binom{0}{A}\binom{0}{3-A}}}\, \eta_{\binom{0}{3-A}}$$

$$\delta_{(\eta - C)}\, d^B, \tag{1.17}$$

Variations of (1.14) in terms of p_a with consideration of the arbitrariness of $\eta(t_1, t_2)$ yields equation

$$j^{\binom{i}{A}\binom{0}{3-A}} = \frac{\partial^2 j}{\partial \tau_{\binom{i}{A}}\, \partial \eta_{\binom{0}{3-A}}}, \quad \frac{\partial^2 j}{\partial \tau_{\binom{i}{A}}\, \partial \eta_{\binom{0}{A}}} = 0,\ A = 1,\ 2. \tag{1.18}$$

We will introduce the definitions:

$$\rho = j^{\binom{0}{1}\binom{0}{2}}, \quad \rho\sigma_1^\alpha = j^{\binom{\alpha}{1}\binom{0}{2}}, \quad \rho\sigma_2^\alpha = j^{\binom{0}{1}\binom{\alpha}{2}}. \tag{1.19}$$

Variation in terms of j^{ab} with consideration of the arbitrariness of $\eta(t_1, t_2)$ yields the relation

$$P_{\binom{\alpha}{A}} = m_A\sigma_A^\alpha + \frac{e_A}{c} A_\alpha(q_A),\ A = 1,\ 2,\ \alpha = 1,\ 2,\ 3, \tag{1.20}$$

$$P_{\binom{0}{A}} = -m_A\, \frac{\sigma_A^\alpha\sigma_A^\alpha}{2} + \frac{\hbar^2}{2m_A}\, \frac{1}{\sqrt{\rho}}\, \frac{\partial^2\sqrt{\rho}}{\partial q_A^\alpha\, \partial q_A^\alpha} + \frac{e_A}{c}\, A_0(q_A). \tag{1.21}$$

It can be said of relations (1.20) and (1.21) that they are derived from the corresponding relations for neutral particles by means of the substitution /12

$$p_{\binom{i}{A}} \to p_{\binom{i}{A}} + \frac{e_A}{c} A_i(q_A).$$

We will renumber all $\mathfrak{z}^B_\alpha$ with the same index $n = 1, 2, \ldots 6S + 2$ ($\mathfrak{z}_1 = \tau$, $\mathfrak{z}_2 = \eta$). Variation in terms of $\mathfrak{z}_n$ ($n - 3, 4, \ldots 6s + 2$) yields the equation

$$\frac{\delta S}{\delta \mathfrak{z}_n} = \sum_{A,B=1}^{2} \delta(\eta - C) \frac{\partial}{\partial x^\alpha} \left(p_{\binom{i}{A}} \eta_{\binom{0}{B}} \frac{\partial^3 j}{\partial \tau_{\binom{i}{A}} \partial \eta_{\binom{0}{B}} \partial \mathfrak{z}_{n,a}} \right) = 0, \tag{1.22}$$

$$n = 3, 4, \ldots 6s + 2.$$

for $n = 1,2$ (1.22) is an identity and therefore is valid for $n = 1, 2, \ldots 6s + 2$.

We will assume for brevity

$$j^{ab} = \frac{\partial^2 j}{\partial \tau_a \partial \eta_b} \tag{1.23}$$

Using the identity

$$\frac{\partial}{\partial x^a} \frac{\partial^3 j}{\partial \tau_c \partial \eta_b \partial \mathfrak{z}_{n,a}} = 0, \tag{1.24}$$

$$\sum_{n=1}^{6S+2} \frac{\partial^3 j}{\partial \tau_c \partial \eta_b \partial \mathfrak{z}_{n,a}} \mathfrak{z}_{n,d} = \delta^a_d j^{cb} + \delta^b_d j^{ac} + \delta^c_d j^{ba}, \tag{1.25}$$

we transform (1.22) to the form

$$\eta_b j^{cb} f_{ac} + 1/2\, \eta_a j^{cb} f_{cb} = 0, \quad a, b, c, - 1, 2, \ldots 8, \tag{1.26}$$

where

$$f_{ac} = \partial_a p_c - \partial_c p_a \qquad a, c = 1, 2, \ldots 8. \tag{1.27}$$

After the transformations (1.26) acquires the form

$$\frac{\partial p_{\binom{\alpha}{A}}}{\partial t_B} = \frac{\partial p_{\binom{0}{B}}}{\partial q_A^\alpha} + \sigma_B^\beta \left(\frac{\partial p_{\binom{\beta}{B}}}{\partial q_A^\alpha} - \frac{\partial p_{\binom{\alpha}{A}}}{\partial q_B^\beta} \right), \quad A, B = 1, 2.$$

Equation (1.26) yields the solution /13

$$f_{ac} = \partial_a p_c - \partial_c p_a = 0, \tag{1.28}$$

although (1.28) does not necessarily follow from (1.26). It follows from (1.28) that

$$p_a = \frac{\partial \varphi}{\partial x^a}, \quad a = 1, 2, \ldots 8, \tag{1.29}$$

where φ is some, as yet arbitrary, function of x.

We will examine the case of a potential solution of (1.29) substituting (1.29) into (1.20), (1.21) and discarding σ_A^α, now we obtain

$$\frac{\partial \varphi}{\partial t_A} + \frac{1}{2m_A} \left[\frac{\partial \varphi}{\partial q_A^\alpha} - \frac{e_A}{c} A_\alpha(q_A) \right] \left[\frac{\partial \varphi}{\partial q_A^\alpha} - \frac{e_A}{c} A_\alpha(q_A) \right] - \tag{1.30}$$

$$- \frac{h^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A^\alpha \partial q_A^\alpha} - \frac{e_A}{c} A_0(q_A), \qquad A = 1, 2.$$

The identity

$$\frac{\partial}{\partial x^b} \frac{\partial^2 j}{\partial \tau_b \partial \eta_{\binom{0}{A}}} = 0 \tag{1.31}$$

is written with the aid of (1.18), (1.19), (1.20) and (1.29) to form equation in the form

$$\frac{\partial \rho}{\partial t_A} + \frac{\partial}{\partial q_A^\alpha}\left[\frac{\rho}{m_A}\frac{\partial \varphi}{\partial q_A^\alpha} - \frac{e_A}{m_A c}\rho A_\alpha(q_A)\right] = 0,\ A - 1,\ 2. \tag{1.32}$$

The values $j^{\binom{\alpha}{1}\binom{\beta}{2}}$ remain undefined. They can be determined by the relation

$$j^{\binom{\alpha}{1}\binom{\beta}{2}} = \rho\sigma_1^\alpha\sigma_2^\beta - \frac{\bar{h}^2}{4m_1m_2}\left(\frac{\partial^2\rho}{\partial q_1^\alpha \partial q_2^\beta} = \frac{1}{\rho}\frac{\partial\rho}{\partial q_1^\alpha}\ \frac{\partial\rho}{\partial q_2^\beta}\right). \tag{1.33}$$

Then in the absence of electromagnetic fields the laws of conservation will be satisfied:

$$\frac{\partial j^{ab}}{\partial \chi^a} = 0,\ a,\ b = 1,\ 2, \ldots 8, \tag{1.34}$$

in the case, however, when $A_i(q) \neq 0$, (1.34), generally speaking, is not satisfied. /14

We will multiply (1.30) by $-\sqrt{\rho}\ \exp\ (i\varphi/\bar{h}$, and (1.32) by $i\bar{h}\ \exp(i\varphi/h)/$ $/(2\sqrt{\rho})$ and combine them. We obtain

$$\left(ih\frac{\partial}{\partial t_A} + \frac{e_A}{c}A_0(q_A)\right)\psi \,\bar{\overline{0}}\, \frac{1}{2m_A}\left(ih\frac{\partial}{\partial q_A^\alpha} + \frac{e_A}{c}A_\alpha\ (q_A)\right) \times \tag{1.35}$$

$$\times\left(ih\frac{\partial}{\partial q_A^\alpha} + \frac{e_A}{c}A_\alpha(q_A)\right)\psi = 0, \quad A = 1,\ 2,$$

where

$$\psi = \sqrt{\rho}\ e^{\frac{i\varphi}{\bar{h}}} \tag{1.36}$$

It is obvious that the two equations (1.35) are always compatible. The equivalent equations (1.30) and (1.32) are also always compatible.

Equations (1.35) describe evolution of the function ψ simultaneously in terms of two times t_1 and t_2. We will now consider the non-relativistic point of view, i.e., we will discuss the behavior of the set at equal times $t_1 = t_2$, i.e., in seven-dimensional plane P_7 of space V_{12}. Carrying out the transformation

$$t = \frac{t_1 + t_2}{2}, \quad \tau = \frac{t_1 - t_2}{2}, \tag{1.37}$$

we arrive, instead of (1.35) at two equations

$$i\hbar \frac{\partial \psi}{\partial t} + \sum_{A=1}^{2} \left\{ \frac{e_A}{c} A_0(q_A) + \frac{\hbar^2}{2m_A} \left(\frac{\partial}{\partial q_A^\alpha} - \frac{ie_A}{\hbar c} A_\alpha(q_A) \right) \times \right.$$

$$\left. \times \left(\frac{\partial}{\partial q_A^\alpha} - \frac{ie_A}{\hbar c} A_\alpha(q_A) \right) \right\} \psi = 0, \tag{1.38}$$

$$i\hbar \frac{\partial \psi}{\partial \tau} + \sum_{A=1}^{2} (-1)^{A-1} \left\{ \frac{e_A}{c} A_0(q_A) + \frac{\hbar^2}{2m_A} \left(\frac{\partial}{\partial q_A^\alpha} - \frac{ie_A}{\hbar c} A_\alpha(q_A) \right) \times \right.$$

$$\left. \times \left(\frac{\partial}{\partial q_A^\alpha} - \frac{ie_A}{\hbar c} A_\alpha(q_A) \right) \right\} \psi = 0. \tag{1.39}$$

Equation (1.38) is the Schrodinger equation for two particles in an external /15
electromagnetic field. It contains τ as a parameter. If the function ψ is known for $t = 0$, $\tau = 0$, it can be determined for any t and $\tau = 0$ by means of only one equation (1.38).

The state of the system in plane P_7 is depicted by a line and not by a two-dimensional surface. Therefore the density of states is depicted by the vector j^i, $(i = 0, 1, \ldots 6)$. In the coordinate system $y^0 = t$, $y^i = q_A^\alpha$,

j^i has the form

$$j^i = \{\rho, \rho v_1^\alpha, \rho v_2^\alpha\}. \tag{1.40}$$

This can be proved by the method employed in [2]. From the laws of conservation (1.32) follows the law of conservation

$$\sum_{i=0}^{6} \frac{\partial}{\partial y^i} j^i = 0. \tag{1.41}$$

It follows from (1.40) that with the appropriate standardization j^0 is the density of the probability of detecting the first particle at point $\vec{q}_1$, and the second particle at point $\vec{q}_2$. The other components denote the probability flow density. They are selected through the wave function so that this is prescribed by the equations of quantum mechanics.

(4'')

2. Set of Interacting Particles /16

We will consider now the case of two charged interacting particles in the absence of an external field. This means that in action (1.2) the fore-potential acting on the first particle is governed by the second particle and conversely. Strictly speaking, we should take into account the degrees of freedom related to the electromagnetic field. I considered only the non-relativistic case, where radiation is completely ignored.

In determining the Lagrangian of a system of two interacting particles the fore-potential A_i in (1.11) should be considered as governed by the charges of the particles. It is also necessary to consider the term omitted in (1.11) that describes the free electromagnetic field. In view of the Maxwell equations it may be written in the form of equation

$$-\frac{1}{16\pi}\int (\partial_i A_k - \partial_k A_i)(\partial^i A^k - \partial^k A^i)\, d^4 q = -\frac{1}{2}\sum_{A=1}^{2}\int e_A A_i(q_A)\frac{dq_A^i}{d\tau}\, d\tau. \tag{2.1}$$

In consideration of this term and the Maxwell equations action (1.11) in non-relativistic approximation ($c \to \infty$) acquires the form

$$S = \sum_{A=1}^{2} \int \left(\frac{m_A \dot q_A^\alpha \dot q_A^\alpha}{2\dot q_A^0} - \frac{e_1 e_2}{2R_{12}} \dot q_A^0 \right) d\tau , \qquad \dot q_A^i \equiv \frac{dq_A^i}{d\tau}, \tag{2.2}$$

$$R_{12} = \sqrt{(q_1^\alpha - q_2^\alpha)(q_1^\alpha - q_2^\alpha)} . \tag{2.3}$$

We will consider a simple set, consisting of systems described by action (2.2). Let $\mathfrak{Z} = \{\mathfrak{z}_1^1, \mathfrak{z}_2^1, \mathfrak{z}_3^1, \mathfrak{z}_1^2, \mathfrak{z}_2^2, \mathfrak{z}_3^2\}$ enumerate the systems of the set and $q_A^i = q_A^i(\tau, \mathfrak{z}, \eta)$, where η is a parameter acquiring the same value of c for all systems in the set. Then we have the action /17

$$S[q_A^i] = \sum_{A=1}^{2} \int \left(\frac{m_A}{2} \frac{\frac{\partial q_A^\alpha}{\partial \tau}\frac{\partial q_A^\alpha}{\partial \tau}}{\frac{\partial q_A^0}{\partial \tau}} - \frac{e_1 e_2}{2R_{12}} \frac{\partial q_A^0}{\partial \tau} \right) \delta(\eta - c)\, d\eta\, d\tau\, d^6\mathfrak{z}, \qquad d^6\mathfrak{z} = \prod_{A=1}^{2}\prod_{\alpha=1}^{3} d\mathfrak{z}_\alpha^A . \tag{2.4}$$

We will transform relations $q_A^i = q_A^i(\tau, \eta, \mathfrak{z})$ and will now regard (2.4) as the functional of τ, $\mathfrak{z}$, $\eta = \tau, \mathfrak{z}, \eta\, (q_A^i)$. Its extremals can be found by varying the action

$$S = S[\mathfrak{z}_\alpha^B] = \sum_{A=1}^{2} \int \left(\frac{m_A}{2} \frac{j^{\binom{\alpha}{A}a} j^{\binom{\alpha}{A}b} \eta_a \eta_b}{j^{\binom{0}{A}c}\eta_c} - \frac{e_1 e_2}{2R_{12}} j^{\binom{0}{A}b} \eta_b \right) \delta(\eta - c)\, d^8x , \tag{2.5}$$

where

$$j^{ab} = \frac{\partial^2 J}{\partial \tau_a \partial \eta_b}, \qquad J = \frac{\partial(\tau, \mathfrak{z}_1^1, \mathfrak{z}_2^1, \mathfrak{z}_3^1, \eta, \mathfrak{z}_1^2, \mathfrak{z}_2^2, \mathfrak{z}_3^2)}{\partial(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8)}, \tag{2.6}$$

and x^a as given by the relation (1.2).

Equation (2.5) is the action for a set of particles interacting according to Coulomb's Law. We will make some generalization in the sense of conversion from (2.6) to (1.8). We will compare (2.5) with (1.6). Then, considering (1.3) and (1.13), we conclude that particle interaction is described by the term

$$S_{12} = -\sum_{A=1}^{2} \int \frac{e_1 e_2}{2R_{12}} j^{\binom{0}{A}\binom{0}{3-A}} \eta_{\binom{0}{3-A}} \delta(\eta - C)\, d^4x . \tag{2.7}$$

Thus, a quantum set of two non-relativistic particles interacting by Coulomb's law is described by the action

$$S = S[j^{ab}, p_a, \mathfrak{z}^{B}_{\alpha}] = S_m + S_{12} + S_{qu}, \tag{2.8}$$

where S_m, S_{12}, S_{qu} are defined by expressions (1.15), (2.7) and (1.17), respectively. /18

Variation in terms of p_a and $\mathfrak{z}^{\alpha}_{B}$ yields the former equations: (1.18) and (1.22), respectively. Variations in terms of $j^{\binom{i}{A}\binom{0}{3-A}}$ yields equations

$$P_{\binom{\alpha}{A}} = m_A v^{\alpha}_A , \quad A = 1,2 , \tag{2.9}$$

$$P_{\binom{0}{A}} = -m_A \frac{v^{\alpha}_A v^{\alpha}_A}{2} + \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q^{\alpha}_A \partial q^{\alpha}_A} - \frac{e_1 e_2}{2R_{12}} , \quad A = 1, 2. \tag{2.10}$$

Further, repeating all calculations from (1.18), (1.20)-(1.22) to (1.35), we obtain, instead of (1.35),

$$\left(i\hbar\frac{\partial}{\partial t_A} - \frac{e_1 e_2}{2R_{12}}\right)\psi + \frac{\hbar^2}{2m_A}\frac{\partial^2\psi}{\partial q_A^\alpha\,\partial q_A^\alpha} = 0, \quad A = 1, 2. \qquad (2.11)$$

Both equations (2.11) are compatible, since equation

$$\frac{\partial^2}{\partial q_A^\alpha\,\partial q_A^\alpha}\frac{1}{R_{12}} = 0 \quad \text{для} \quad q_1^\alpha \neq q_2^\alpha. \qquad (2.12)$$

We obtain, instead of (1.38) and (1.39),

$$i\hbar\frac{\partial\psi}{\partial t} - \frac{e_1 e_2}{R_{12}} + \sum_{A=1}^{2}\frac{\hbar^2}{2m_A}\frac{\partial^2\psi}{\partial q_A^\alpha\,\partial q_A^\alpha} = 0, \qquad (2.13)$$

$$i\hbar\frac{\partial\psi}{\partial\tau} + \sum_{A=1}^{2}(-1)^{A-1}\frac{\partial^2\psi}{\partial q_A^\alpha\,\partial q_A^\alpha} = 0. \qquad (2.14)$$

Equation (2.13) is the Schroedinger equation for two non-relativistic particles interacting by Coulomb's law.

Finally, the action for a quantum set of interacting particles in an external electromagnetic field is written in the form

$$S = S[j^{\alpha l}, p_a, \mathfrak{z}^\beta_a] = S_m + S_{12} + S_{m\gamma} + S_{12}, \qquad (2.15)$$

where S_m, S_{12}, $S_{m\gamma}$, S_{qu} are given by (1.15), (2.7), (2.16) and (1.17), /19
respectively. It is obvious that the Schrodinger equation for two interacting non-relativistic particles in an external electromagnetic field can be extracted from the equations of motion for such a set.

3. Energy, Motion, and Moment of Quantum Set

To a quantum set, by any dynamic system, can be made to correspond to energy, motion and moment. These values can be determined cannonically from the Lagrangian. Let action (2.15) be defined as the integral for some region Ω of space V_{12}:

$$S = \int_{\Omega} L\, d^8x \tag{3.1}$$

We will subject coordinate x_a to infinitesimally small transformation

$$x^a \to x^a + \delta x^a. \tag{3.2}$$

In the case when δx^0 = const transformation (3.2) enduces variation of action of the form

$$\delta S = -\int_{\Sigma} T_b^{ac}\, \delta x^b \eta_c\, \delta(\dot{\eta} - C)\, dS_a, \tag{3.3}$$

where Σ is the seven-surface bounding space Ω, and dS_a is an element of this surface. Here

$$T_b^{ac}\, \eta_c\, \delta(\eta - \overset{\circ}{C}) = \sum_{\gamma} \frac{\partial L}{\partial u_{\gamma,a}} u_{\gamma,b} - \delta_b^a L, \tag{3.4}$$

where

$$u_\gamma = \{ j^{ab}, p_a, \xi_\alpha^\beta, \eta \}, \qquad u_{\gamma,a} \equiv \frac{\partial u_\gamma}{\partial x^a}, \tag{3.5}$$

and summation is done in terms of all indices that enumerate the variables that /20 have to be changed, including η, the fact that the left-hand side of (3.4) can be written as the convolution of $T_b^{ac}\,\eta_c$ is the result of the specific form of Lagrangian determined by relations (1.15), (2.7), (1.16) and (1.17).

In the case when base Ω is bounded by two surfaces $t_1 = T_1 = \text{const}$ and $t_1 = T_2 = \text{const}$ ($T_2 \quad T_1$), by selecting $\eta(t_1, t_2) = t_2 - t_1$, we obtain for (3.3)

$$\delta S = -\int\limits_{t_1 = t_2 = T_2} T_b^{(\overset{0}{1})(\overset{0}{2})}\,\delta x^b\, d\vec{q}_1\, d\vec{q}_2 + \int\limits_{t_1 = t_2 = T_1} T_b^{(\overset{0}{1})(\overset{0}{2})}\,\delta x^b\, d\vec{q}_1\, d\vec{q}_2\,, \qquad (3.5)$$

$$d\vec{q}_A = dq_A^1\, dq_A^2\, dq_A^3\,, \qquad A = 1, 2.$$

Vector

$$\mathcal{P}_b = \int T_b^{(\overset{0}{1})(\overset{0}{2})}\, d\vec{q}_1\, d\vec{q}_2 = \int T_b^{15}\, d\vec{q}_1\, d\vec{q}_2\,; \qquad b = 1, 2, \dots 8. \qquad (3.6)$$

plays a part

(4'')

plays the part of the energy-motion vector and remains valid for a set of free particles. T_b^{ac} plays the part of the energy-motion tensor. The fact that this tensor is of the third order and not the second, as is usually the case, is related to the presence of two times.

Calculation by equation (3.4) yields for the energy-motion density of the system, described by action (2.15),

$$T_{(\overset{\cdot}{A})}^{A5} = -p_{(\overset{\cdot}{A})}\, \rho\,, \qquad (3.7)$$

where $p_{(\overset{\alpha}{A})}$ is given by relation (1.20), and equation

$$p_{(\overset{0}{A})} = -m_A \frac{v_A^\alpha v_A^\alpha}{2} + \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A^\alpha \partial q_A^\alpha} + \frac{\epsilon_A}{c} A_0(q_A) - \frac{\epsilon_1 \epsilon_2}{R_{12}} \qquad (3.8)$$

We will raise the lower index in (3.7) with the aid of five-dimensional metric tensor (see Appendix). We obtain in gauge-invariant form /21

$$T^{15,\binom{\alpha}{A}} = m_A v_A^\alpha \rho, \tag{3.9}$$

$$c^2 T^{15,\binom{0}{A}} = \left(\frac{m_A v_A^\alpha v_A^\alpha}{2} - \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A^\alpha \partial q_A^\alpha} + \frac{e_1 e_2}{2 R_{12}} \right) \rho. \tag{3.10}$$

Examining in like fashion transformation (3.2), which describes infinitestimally small rotation in the plane t_A = const of space V_A, we may introduce the moment of motion:

$$M_A^{\alpha\beta} = \int M^{15,\binom{\alpha}{A}\binom{\beta}{A}} d\vec{q}_1 d\vec{q}_2, \tag{3.11}$$

where in the given case

$$\begin{aligned} M^{15,\binom{\alpha}{A}\binom{\beta}{B}} &= \delta_{AB} \left\{ q_B^\beta T^{15,\binom{\alpha}{A}} - q_A^\alpha T^{15,\binom{\beta}{A}} \right\} = \\ &= \delta_{AB} \left\{ q_A^\beta v_A^\alpha - q_A^\alpha v_A^\beta \right\} m_A \rho. \end{aligned} \tag{3.12}$$

We introduce the operators

$$\hat{P}_{\binom{\alpha}{A}} = -i\hbar \frac{\partial}{\partial q_A^\alpha}, \qquad \hat{p}^{\binom{\alpha}{A}} = i\hbar \frac{\partial}{\partial q_A^\alpha} + \frac{e_A}{c} A_\alpha(q_A). \tag{3.13}$$

We will assume that condition (1.28) is satisfied. Then

$$\mathcal{P}^{(\alpha)} = -\int \psi^* \hat{p}^{(\alpha)} \psi \, d\vec{q}_1 \, d\vec{q}_2 , \tag{3.14}$$

$$c^2 \mathcal{P}^{(0)} = \int \psi^* \left(\frac{\hat{p}^{(\alpha)} \hat{p}^{(\alpha)}}{2 m_A} + \frac{e_1 e_2}{2 R_{12}} \right) \psi \, d\vec{q}_1 \, d\vec{q}_2 , \tag{3.15}$$

where ψ^* is a value complex conjugate to ψ, which is defined, in turn by

$$M_A^{\alpha\beta} = \int \psi^* (q_A^{\alpha} \hat{p}^{(\beta)} - q_A^{\beta} \hat{p}^{(\alpha)}) \psi \, d\vec{q}_1 \, d\vec{q}_2 . \tag{3.16}$$

In the case when the particles are uncharged ($e_1 = e_2 = 0$), all values $P^{\binom{\alpha}{A}}$, $c^2 P^{\binom{0}{A}}$, $M_A^{\alpha\beta}$ remain in force. Since $P^{\binom{\alpha}{A}}$, $c^2 P^{\binom{0}{A}}$ (4''), $M_A^{\alpha\beta}$ (A = 1, 2) are /22 additives and related respectively to spatial displacement, temporal displacement and spatial rotation, then according to the statistical principle they can be regarded respectively as the mean motion of the eighth particle, mean energy of the eighth particle and mean moment of the eighth particle.

Equations (3.14)-(3.16) coincide with the rule of calculating the means of these values in quantum mechanics if ψ is defined by the relation

$$\int \psi^* \psi \, d\vec{q}_1 \, d\vec{q}_2 = 1 . \tag{3.17}$$

When this condition is satisfied in view of definition (1.19), $\rho = \psi^*\psi$ can be regarded as the density of the probability of detecting the first particle at point $\vec{q}_1$ and the second particle at point $\vec{q}_2$. For this reason the mean value of the arbitrary function $F(\vec{q}_1, \vec{q}_2)$ is determined by the relation

$$\langle F \rangle = \int \psi^* F(\vec{q}_1, \vec{q}_2)\, \psi \, d\vec{q}_1 \, d\vec{q}_2 . \qquad (3.18)$$

The brackets denote the mean value.

4. Stationary States of Quantum Sets and Their Significance /23

We will consider a quantum set of two interacting particles in a given external magnetic field. Let the electromagnetic field be stationary. Then the fore-potential A_i may also be made stationary, i.e.,

$$\frac{\partial A_i(q_A)}{\partial t_A} = 0, \qquad A_i(q_A) = A_i(\vec{q}_A). \qquad (4.1)$$

The state of the set depends, generally speaking, on two times t_1 and t_2, or in variables (1.37), on t and τ. We will analyze the set for identical times $t_1 = 2_2$ or for $\tau = 0$, which is equivalent. We will call the state of the set stationary if it does not depend on t when $\tau = 0$, i.e.,

$$\frac{\partial j^{ab}}{\partial t} = 0, \qquad \frac{\partial \rho_a}{\partial t} = 0 \quad \text{where} \quad \tau = 0, \qquad a, b = 1, 2, \dots 8. \qquad (4.2)$$

Conditions (4.2), in fact, are not independent, in the second, in the view of (1.19), (1.20), (3.8) and (4.1), is the consequence of the first condition (4.2).

We will find an equation which the stationary state satisfies in the assumption ahat (1.28) is satisfied. From (4.2) and (1.29) follows

$$\varphi = \varphi_0(\vec{q}_1, \vec{q}_2) + \varphi_{10}(t) \qquad (4.3)$$

(The function of τ is not indicated.) Adding to equation (1.30) the term

$-\varepsilon_1\varepsilon_2/(2R_{12})$, we write them in the form

$$\frac{\partial\varphi}{\partial t} = \sum_{A=1}^{2}\left\{-\frac{1}{2m_A}\left[\frac{\partial\varphi}{\partial q_A^\alpha} - \frac{e_A}{c}A_\alpha(q_A)\right]\left[\frac{\partial\varphi}{\partial q_A^\alpha} - \frac{e_A}{c}A_\alpha(q_A)\right] + \right. \\ \left. + \frac{\hbar^2}{2m_A}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial q_A^\alpha \partial q_A^\alpha} + \frac{e_A}{c}A_0(q_A)\right\} - \frac{\varepsilon_1\varepsilon_2}{R_{12}}. \tag{4.4}$$

The right-hand side does not depend on t, therefore, $\partial\psi/\partial t$ also does not depend on t, and (4.3) acquires the form /24

$$\varphi = \varphi_0(\vec{q}_1, \vec{q}_2) - H't, \tag{4.5}$$

where H' is a real constant. Combining the two equations (1.32) and considering ρ to be independent of t, we obtain

$$\sum_{A=1}^{2}\frac{\partial}{\partial q_A^\alpha}\left\{\frac{\rho}{m_A}\frac{\partial\varphi}{\partial q_A^\alpha} - \frac{\varepsilon_A}{m_A c}\rho A_\alpha(\vec{q}_A)\right\} = 0. \tag{4.6}$$

Combining (4.4) and (4.6) we obtain for the (4'') function ψ from (1.36) the equation

$$\sum_{A=1}^{2}\left\{-\frac{\hbar^2}{2m_A}\left(\frac{\partial}{\partial q_A^\alpha} - \frac{i\varepsilon_A}{\hbar c}A_\alpha(\vec{q}_A)\right)\left(\frac{\partial}{\partial q_A^\alpha} - \frac{i\varepsilon_A}{\hbar c}A_\alpha(\vec{q}_A)\right) - \right. \\ \left. - \frac{\varepsilon_A}{c}A_0(\vec{q}_A)\right\}\psi + \frac{\varepsilon_1\varepsilon_2}{R_{12}}\psi = H'\psi. \tag{4.7}$$

Thus, the problem of finding the stationary state of the set is reduced to the problem of seeking out the Eigenfunctions and corresponding Hamiltonians:

$$\hat{H} = \sum_{A=1}^{2}\left\{\frac{1}{2m_A}\hat{P}^{(\alpha)}\hat{P}^{(\alpha)} - \frac{\varepsilon_A}{c}A_0(\vec{q}_A)\right\} + \frac{\varepsilon_1\varepsilon_2}{R_{12}}. \tag{4.8}$$

It is obvious that the converse is also valid, i.e., if

$$\psi_0(\vec{q}_1, \vec{q}_2, t) = e^{-\frac{iH't}{\hbar}}\psi_0(\vec{q}_1, \vec{q}_2), \tag{4.9}$$

where $\psi_0(\vec{q}_1, \vec{q}_2)$ is Eignefunction $\hat{H}$ with eigenvalues H', then the values j^{ab}

constructed from ψ will not depend on t. Actually j^{ab} can be constructed from v_A^α and ρ by using equations (1.19) and (1.33). For ρ and v_A^α we have

$$\rho = \psi_0^*(\vec{q}_1, \vec{q}_2)\,\psi_0(\vec{q}_1, \vec{q}_2), \tag{4.10}$$

$$v_A^\alpha = \frac{2i\hbar}{m_A} \frac{\partial}{\partial q_A^\alpha} \ln \frac{\psi_0^*(\vec{q}_1, \vec{q}_2)}{\psi_0(\vec{q}_1, \vec{q}_2)} + \frac{e_A}{c} A_\alpha(\vec{q}_A). \tag{4.11}$$

i.e., ρ and v_A^α are not functions of t. /25

The traditional statistical interpretation of quantum mechanics [4, chapter 3, section 1] can be derived from the following two hypotheses.

1. If to R corresponds operator $\hat{R}$, then to f(R) corresponds operator $f(\hat{R})$.

2. The mean of any value of R in state ψ is defined by the relation

$$\langle R \rangle = \int \psi^* \hat{R} \psi \, dV. \tag{4.12}$$

The integral in (4.12) denotes integration in terms of all arguments on which the wave function depends.

The validity of (4.12) was derived from relativistic statistics[5] only for the additive values and arbitrary functions of the coordinates. Relation (4.12) for the arbitrary value R cannot be derived from relativistic statistics. Moreover (4.12) is incompatible with relativistic statistics, since it follows from (4.12) that a particle cannot possess simultaneously a certain coordinate and a certain pulse [5]. In this connection the following question arises: to what degree is (4.12) essential for explaining experimental data and is it possible to explain experimental data simply on the basis of relativistic statistics? I cannot answer this question conclusively here and will make only a few comments.

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[5] I call relativistic statistics the concept advanced in [1, 2] and developed in this article.

It follows from (4.12) that measurements can given for R only a value coinciding with one of the eigenvalues of operator $\hat{R}$, corresponding to R. The /26 fact is, however, that it is possible to measure only those values which commute with the Hamiltonian of the system, and the state of the system being measured must be stationary. This was proved by Von Neuman [4, Chapter 5, section 1).

Actually, in the framework of quantum mechanics measurement of any value R pertaining to system S, with wavefunction ψ, amounts to some action on system S. As a result of this action, Hamiltonian $\hat{H}$ of the system is measured so that the values of R begin to commute with operator $\hat{R}$, and state ψ becomes a stationary state, i.e., the Eigenstate of operator $\hat{H}$ of the system. This occurs because no measurement is made instantaneously and state ψ must be such that it changes little during the time of measurement, i.e., should be the stationary state. But if operator $\hat{R}$ commutes with the Hamiltonian its Eigenvalues R* may be used for numbering the Eigenstate of the Hamiltonian.

Relativistic statistics states, on the other hand, that the stationary states can be found as the Eigenstates of the Hamiltonian. This was proved for the case of two interacting particles in an electromagnetic field, and is apparently valid for other cases. Therefore, the R' of any measured value R can be regarded as the "number" of its stationary state, and it can be determined by identifying the stationary state. From this point of view any measurement can be reduced to identification of the stationary state of a quantum set. The stationary states here play an exceptionally important role.

Suppose, for instance, an atom is placed in a magnetic field directed /27 along the c-axis. Operator $\hat{M}_z$ of the projection of the moment onto the z-axis commutes with the Hamiltonian of the atom and the energy levels are numbered by the Eignevalues of operator $\hat{M}_z$ (but not by them alone). Suppose the atom, under the influence of excitation, changes from one stationary state ψ' with $M_z = M_z'$ to another stationary state ψ'' with $M_z = M_z''$ and emits a photon. By recording the frequency of the photon it is possible to identify the levels between which transition occurred and to determine M_z' and M_z''.

Thus, in the given case the moment of motion M_z is measured only to the extent that it numbers the stationary state. If it can be shown that any real measurement amounts to identification of some stationary state, it thereby will be shown that relativistic statistics can explain experimental facts just as successfully as quantum mechanics.

If two particles are identical, then their identity is considered as is done in [2], leading to the relation

$$\psi(t, \vec{q}_1, \vec{q}_2) = \psi(t, \vec{q}_2, \vec{q}_1). \tag{4.13}$$

Generalization of all results to the case of n interacting particles is an easy task.

APPENDIX /28

GAUGE-INVARIANT FORM OF ENERGY-MOTION TENSOR FOR PARTICLE IN ELECTROMAGNETIC FIELD

The motion of a particle in an electromagnetic field is described by the action of

$$S = S_m + S_e = \int L \sqrt{-g}\, d^4x, \tag{A.1}$$

$$S_m = S_m[q^i(\tau), A_k(q)] = \int \left\{ -mc\sqrt{\dot{q}^i g_{ik} \dot{q}^k} + \frac{e}{c} A_i(q)\dot{q}^i \right\} d\tau,$$

$$S_e = S_e[A_k(x)] = -\frac{1}{16\pi}\int F_{ik} F^{ik} \sqrt{-g}\, d^4x, \tag{A.2}$$

$$F_{ik} = F_{ik}(x) = \partial_i A_k(x) - \partial_k A_i(x), \tag{A.3}$$

where x^i are arbitrary curvilinear coordinates in the prime space, q_{ik} is the metric tensor, and

$$g = \det \| g_{ik} \|. \tag{A.4}$$

The energy-motion tensor can be calculated by two different means. The first means, variation in terms of q_{ik}, yields equation

$$T^{\cdot ik}(x) = -\frac{\delta S}{\delta g_{ik}(x)} = -\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{ik}(x)} (\sqrt{-g}\, L). \tag{A.5}$$

The second, cannonical, yields

$$\Theta^i_{\ k} = \sum_{\gamma} \frac{\partial L}{\partial u_{\gamma,i}} u_{\gamma,k} - \delta^i_k L, \tag{A.6}$$

where u_γ are variables, in terms of which the action is varied for obtaining the equations of motion.

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The first method yields, respectively, for S_m and S_e /29

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$$T_m{}^{ik}(x) = mc \frac{\dot q^i(\tau_0)\, \dot q^k(\tau_0)}{\sqrt{\dot q^j(\tau_0)\, g_{js}(x)\, \dot q^s(\tau_0)}} \; \frac{\delta(\vec q(\tau_0) - \vec x)}{|\dot q^0(\tau_0)|}, \tag{A.7}$$

and τ_0 is the root of the equation

$$q^0(\tau_0) - x^0 = 0, \tag{A.8}$$

$$T_e{}^{ik}(x) = -\frac{1}{4\pi}\left\{F^{ij}F^k{}_j - \frac{1}{4} g^{ik} F^{js} F_{js}\right\}. \tag{A.9}$$

(4'')

The cannonical method, whereas, yields

$$\Theta_m{}^i{}_k(x) = \left\{\frac{mc\, \dot q^i g_{kl} \dot q^l}{\sqrt{\dot q^j g_{js} \dot q^s}} - \frac{e}{c} A_k(x)\, \dot q^i\right\} \frac{\delta(\vec q - \vec x)}{|\dot q^0|}. \tag{A.10}$$

The argument τ_0 is omitted everywhere

$$\Theta_e{}^i{}_k(x) = g_{kl} T_e{}^{il}(x) - \frac{1}{4\pi}\partial_l (A_k F^{il}) + \frac{1}{4\pi} A_k \partial_l F^{il}. \tag{A.11}$$

From the Maxwell equation

$$\partial_l F^{il} = \frac{4\pi e}{c}\, \dot q^i\, \frac{\delta(\vec q - \vec x)}{|\dot q^0|} \tag{A.12}$$

follows

$$g_{ik}(T_m{}^{ik} + T_e{}^{ik}) = \Theta_m{}^i{}_i + \Theta_e{}^i{}_i + \frac{1}{4\pi}\partial_l (A_k F^{il}). \tag{A.13}$$

Thus the different methods of determining the energy-motion tensor yields the same expression for the complete energy and motion, but the energy is distributed differently between the particles and the electromagnetic field.

If, however, we take the point of view [6]-[9] that the real space time is bi-dimensional and closed with respect to the fifth coordinate x^4, where the fifth coordinate is spacelike, and denotes that the corresponding cannonical /30 pulse p_4 is the electrical charge, expressions (A.7) for T_m^{ik} and (A.10) for $\overline{T}_m^{ik}$ are equivalent. The fact is that in such five-dimensional space the metric tensor γ^{ab}, A,B = 0, 1, 2, 3, 4 has the form

$$\left.\begin{aligned} &\gamma^{ik} = g^{ik}, \qquad \gamma^{i4} = \gamma^{4i} = -g^{ik} A_k Q^{-1}, \\ &\gamma^{44} = -1 + A_i g^{ik} A_k Q^{-2}, \qquad i, k = 0, 1, 2, 3. \end{aligned}\right| \tag{A.14}$$

where Q is some universal dimensional constant (energy × charge^{-1}). The cannonical energy-motion-charge tensor is of the (4'') form $\{\Theta_m{}^i{}_k, \Theta_m{}^i{}_4\}$, and $\Theta_m{}^i{}_k$ is given by relation (A.10), $\Theta_m{}^i{}_4$ describes the fore-current and has the form

$$\Theta_m{}^i{}_4 = \frac{e}{c} \dot{q}^i \frac{\delta(\vec{q} - \vec{x})}{|\dot{q}^0|} Q. \tag{A.15}$$

By raising the second index of $\Theta_m{}^i{}_A$ (A = 0, 1, 2, 3, 4) by means of γ^{AB}, we obtain

$$\Theta_m{}^{ik} = g^{k\ell}\Theta_m{}^i{}_\ell + \gamma^{k4}\Theta_m{}^i{}_4 = T_m{}^{ik}. \tag{A.16}$$

Thus, from the point of view of equations (A.7) and (A.10), these are two different forms of the same expression. (A.7) is gauge-invariant expression and has an advantage over (A.10). In such five-dimensional interpretation, generally speaking, the gauge-invariant tensor components are those whose indices, acquiring the values 0, 1, 2, 3, are contravariants, and those acquiring the value 4 are covariants.

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